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UNIVERSAL EXACTNESS IN ALGEBRAIC K-THEORY

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Introduction

This brief note conveys a simple fact about exactness of K-theory exact sequences: even if they are not split, they tend to be filtering inductive limits of sequences which do split, and so remain exact after tensoring with any abelian group, e.g. $\mathbb{Z}/m$. The primary exact sequence to which this applies is the Gersten resolution of the sheaf $\mathcal{K}_q$ on a nonsingular algebraic variety X. It follows that

$$A^q(X)\otimes\mathbb{Z}/m=H^q(X,\mathcal{K}_q\otimes\mathbb{Z}/m)$$

where A^q denotes algebraic cycles of codimension q modulo linear equivalence.

This fact, for $q=2$, was used by Bloch [1,2] and by Colliot-Thélène, Sansuc, and Soulé [3, p. 778]. The arguments there are special for the lower K-groups.

Results

In the following we use freely the notation from Quillen [6]. In addition, if $f:X\to Y$ is a map of spaces we let $\Omega(f)=\Omega(X\to Y)$ denote the homotopy fiber.

Lemma 1. *Suppose $F,G:\mathcal{M}\to\mathcal{N}$ are exact functors of exact categories $\mathcal{M},\mathcal{N}$, and suppose*

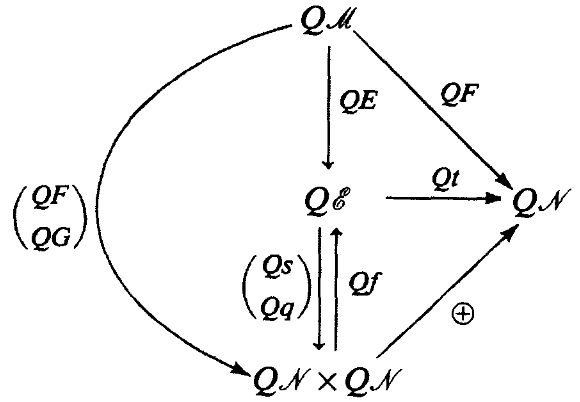
$$E:0\to F\to F\to G\to 0$$

is an exact sequence of functors. The natural map

$$\Omega(BQG)\to BQ\mathcal{M}$$

has a section, or equivalently, the map BQG is null-homotopic.

Proof. It follows immediately from Corollary 1 to Theorem 2 of [6], that $\pi_*BQG=0$, so what we want to show is only slightly stronger, and follows from a brief examination of Quillen's proof. Consider the diagram



Here $\mathcal{E}$ denotes a certain exact category whose objects are short exact sequences in $\mathcal{N}$. The functors $s, t, q : \mathcal{E} \to \mathcal{N}$ pick out the sub-, total, and quotient objects. We define

$$f(N_1, N_2) = (0 \to N_1 \to N_1 \oplus N_2 \to N_2 \to 0).$$

Theorem 2 of [6] says that $\binom{Qs}{Qq}$ is a homotopy equivalence, and Qf is evidently a right inverse to it. Thus Qf is homotopic to any homotopy inverse of $\binom{Qs}{Qq}$, so is a left homotopy inverse as well. Thus

$$\begin{aligned} QF &= Qt \circ QE \\ &\sim Qt \circ Qf \circ \begin{pmatrix} Qs \\ Qq \end{pmatrix} \circ QE \\ &= \oplus \circ \begin{pmatrix} QF \\ QG \end{pmatrix} = QF \oplus QG. \end{aligned}$$

Since $BQ\mathcal{N}$ is a connected H-space under the operation $\oplus$, it has an H-inverse [5], and we may cancel in the equation

$$BQF \sim BQF \oplus BQG$$

to get $* \sim BQG$. □

Corollary 2. *The short exact sequences*

$$0 \to K_* \mathcal{N} \to \pi_* \Omega BQG \to K_{*-1} \mathcal{M} \to 0$$

are split.

Remark. In the proof of the Fundamental Theorem [5] two pages were spent splitting the boundary map $K_q A[t, t^{-1}] \to K_{q-1} A$. Corollary 2 provides a simpler proof that the splitting exists.

Corollary 3. *Suppose we have a filtering inductive limit* $\mathcal{M} = \varinjlim \mathcal{M}_\alpha$, *and* $G_\alpha : \mathcal{M}_\alpha \to \mathcal{N}$ *are compatible exact functors with* $G = \varinjlim G_\alpha : \mathcal{M} \to \mathcal{N}$. *Assume for* *each α that an exact functor $F_\alpha : \mathcal{M}_\alpha \to \mathcal{N}$ exists as well as an exact sequence $0 \to F_\alpha \to F_\alpha \to G_\alpha \to 0$. Then the short exact sequence*

$$0 \to K_*\mathcal{N} \to \pi_*\Omega BQG \to K_{*-1}\mathcal{M} \to 0$$

is a filtering inductive limit of split exact sequences, so remains exact upon application of any additive functor which commutes with filtering inductive limits.

Proof. The constructions Ω, B, Q, π_*, and the notion of exactness all commute with filtering inductive limits. □

Definition 4. An exact sequence of abelian groups is *universally exact* if the application to it of any additive functor which commutes with inductive limits, yields an exact sequence.

Definition 5. An exact sequence of sheaves is *universally exact* if its stalks are.

Remark. The tensor product of a universally exact sequence of sheaves with any fixed abelian sheaf is exact, as can be seen by checking the stalks.

Remark. The main virtue of Godement's canonical resolution of a sheaf [4] is its universal exactness.

Corollary 6. *If X is a nonsingular algebraic variety, then the Gersten–Quillen resolution of the sheaf $\mathcal{K}_q(\mathcal{O}_X)$ is universally exact.*

Proof. Examine the proof of Theorems 5.11 and 5.8 of Quillen [6], to see that Corollary 3 applies to the short exact sequences which, when spliced together and sheafified, form the Gersten–Quillen resolution. □

References

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